ECE 298JA **DE #1 – Version 2.11 October 22, 2016** **Fall 2015**

Univ. of Illinois **Due Mon, Oct 24, 2016** **Prof. Allen**

Topic of this homework: Complex numbers and functions (ordering and algebra); Complex power series; Fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions; Multivalued functions (branch cuts and Riemann sheets)

1 Algebra with complex numbers

1. One can always say that 3<4, namely that real numbers have *order*. One way to view this is to take the difference, and compare to zero, as in $4 - 3 > 0$. Here, we will explore how complex numbers may be ordered. Define $z = x + iy \in \mathbb{C}$.

   (a) Explain the meaning of $|z_1| > |z_2|$.

   (b) If $x_1, x_2 \in \mathbb{R}$ (are *real* numbers), define the meaning of $x_1 > x_2$. *Hint: Take the difference.*

   (c) (*not graded*) If time were complex how might the world be different?

2. It is sometimes necessary to consider a function $w(z) = u + iv$ in terms of the real functions, $u(x, y)$ and $v(x, y)$ (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse $z(w) = x + iy$ where $x(u, v)$ and $y(u, v)$ are real functions.

   (a) Find $u(x, y)$ and $v(x, y)$ for $w(z) = 1/z$.

   (b) Find $u(x, y)$ and $v(x, y)$ for $w(z) = c^z$ with complex constant $c \in \mathbb{C}$ for the following cases

      i. $c = e$

      ii. $c = 1$ (recall that $1 = e^{i2\pi k}$ for $k = 0, 1, 2, \ldots$)

   (c) Find $u(x, y)$ for $w(z) = \sqrt{z}$. *Hint: Begin with the inverse function $z = w^2$.*

2 Complex Power Series

1. In each case derive (e.g. using Taylor's formula) the power series of $w(s)$ about $s = 0$ and state the ROC of your series. If the power series doesn't exist, state why! *Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at $s = 0$.*

   (a) $1/(1 - s)$

   (b) $1/(1 - s^2)$.

   (c) $1/(1 - s)^2$

   (d) $1/(1 + s^2)$. *Hint: This series will be very ugly to derive if you try to take the derivatives $\frac{d^n}{ds^n}[1/(1+s^2)]$. Using the results of our previous homework, you should represent this function as $w(s) = -0.5i/(s - i) + 0.5i/(s + i)$.*

   (e) $1/s$

   (f) $1/(1 - |s|^2)$

2. Consider the function $w(s) = 1/s$

(a) Expand this function as a power series about $s = 1$. What is the ROC?

(b) Expand this function as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$ (same as $s = 1$). What is the ROC? *Hint: Let* $z = s^{-1}$.

3. Consider the function $w(s) = 1/(2 - s)$

   (a) Expand $w(s)$ as a power series in $s^{-1} = 1/s$. State the ROC as a condition on $|s^{-1}|$. *Hint: Let* $z = s^{-1}$.

   (b) Find the inverse function $s(w)$. Where are the poles and zeros of $s(w)$, and where is it analytic?

3 Two fundamental theorems of calculus

Fundamental Theorem of Calculus (Leibniz): According to the Fundamental Theorem of (Real) Calculus (FTC)

$$F(x) = F(a) + \int_a^x f(\xi)d\xi \tag{1}$$

Where $x, a, \xi, F \in \mathbb{R}$. This is known as the *indefinite integral* (since the upper limit is unspecified). It follows that

$$\frac{dF(x)}{dx} = \frac{d}{dx}\int_a^x f(x)dx = f(x).$$

This justifies also calling the indefinite integral the *anti-derivative*.

For a closed interval $[a, b]$, the FTC is often stated as

$$\int_a^b f(x)dx = F(b) - F(a), \tag{2}$$

meaning that the result of the integral is independent of the path from $x = a$ to $x = b$.

Fundamental Theorem of *Complex* Calculus: According to the Fundamental Theorem of Complex Calculus (FTCC)

$$F(z) = F(z_0) + \int_{z_0}^z f(\zeta)d\zeta, \tag{3}$$

where $z_0, z, \zeta, F \in \mathbb{C}$. It follows that

$$\frac{dF(z)}{dz} = \frac{d}{dz}\int_{z_0}^x f(\zeta)d\zeta = f(z).$$

To do:

1. Consider Equation 1. What is the condition on $f(x)$ for which this formula is true?

2. Consider Equation 3. What is the condition on $f(z)$ for which this formula is true?

3. Perform the following integrals ($z = x + iy \in \mathbb{C}$)

   (a) $I = \int_0^{1+\jmath} zdz$

   (b) $I = \int_0^{1+\jmath} zdz$, but this time make the path explicit: from 0 to 1, with y=0, and then to y=1, with x=1.

   (c) Do you results agree with Equation 2?

4. Perform the following integrals on the closed path C, which we define to be the unit circle. You should substitute $z = e^{i\theta}$ and $dz = ie^{i\theta}d\theta$, and integrate from $[-\pi, \pi]$ to go once around the unit circle.

   (a) $\int_C zdz$
   (b) $\int_C \frac{1}{z}dz$
   (c) Do you results agree with Equation 2? If not, do you know why not?

4 Cauchy-Riemann Equations

For the following problem: $i = \sqrt{-1}$, $s = \sigma + i\omega$, and $f(s) = u(\sigma, \omega) + iv(\sigma, \omega)$.

In class I showed that the integration of a complex analytic function is independent of the path, formally known as the *Fundamental theorem of complex calculus.* The derivative of $f(s)$ is defined as

$$\frac{df}{ds} = \frac{d}{ds}\left[u(\sigma, \omega) + \jmath v(\sigma, \omega)\right]. \tag{4}$$

If the integral is independent of the path, then the derivative must also be independent of direction

$$\frac{df}{ds} = \frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial \jmath\omega}. \tag{5}$$

1. The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}$$

   may be used to show where Equation 5 holds.

   (a) Assuming Equation 5 is true, use it to derive the CR equations.
   (b) Merge the CR equations to show that u and v obey Laplace's equation ($\nabla^2 u(\sigma, \omega) = 0$ and $\nabla^2 v(\sigma, \omega) = 0$). One may conclude that the real and imaginary parts of complex analytic functions must obey these conditions.

2. Apply the CR equations to the following functions. State for which values of $s = \sigma + i\omega$ the CR conditions do or do not hold (e.g. where the function $f(s)$ is or is not analytic). *Hint: Recall your answers to the first problem of the assignment!*

   (a) $f(s) = e^s$
   (b) $f(s) = 1/s$

5 Riemann Sheets and Branch cuts

1. Consider the function $[w(z)]^2 = z$. This function can also be written as $w(z) = \pm\sqrt{z}$, making it 'multivalued.'

   (a) How many Riemann sheets do you need in the domain (z) and the range (w) to fully represent this function? Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.
   (b) Use `zviz.m` to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$. Describe what you see.

(c) Where does `zviz.m` place the branch cut for this function? Must it necessarily be in this location?

2. Consider the function $f(z) = \log(z)$.

   (a) Describe with a sketch, and then discuss the *branch cut* for $f(z)$.

   (b) What is the inverse of this function, $z(f)$? Does this function have a branch cut (if so, where is it)?

3. Using zviz, show that

$$\tan^{-1}(z) = -\frac{\jmath}{2} \log \frac{\jmath - z}{\jmath + z}.$$

That is, plot both these function and verify they are the same function, using Matlab commands `atan(Z)` and `-(j/2)*log((j+Z)./(j-Z))`.